Quadratically constrained quadratic programs on acyclic graphs with application to power flow

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Primal Problem

Primal problem $P$:

\[
\begin{align*}
\text{minimize} & \quad x^*C x \\
\text{subject to:} & \quad x^*C_k x \leq b_k, \quad k \in \mathcal{K}.
\end{align*}
\]

G(P) is a tree, i.e. connected and acyclic
Conditions

Condition 1: (a) $G(P)$ is connected and acyclic.
(b) For any edge $(i, j)$ in $G(P)$, the origin is not in the relative interior of the convex hull of $\{C_{ij}, [C_k]_{ij}, k \in K\}$.
(c) The set of feasible solutions of $P$ is bounded and has a strictly feasible point.

These conditions always imply that we can get the global optimal solution in polynomial time.
Condition 1.b.

(a) Origin does not belong to the relative interior of the convex hull of these points.

(b) Origin lies in the relative interior of the convex hull of these points.
Relaxed Problem

Relaxed Problem $RP$:

\[
\begin{align*}
\text{minimize} & \quad \text{tr}(CW) \\
\text{subject to} & \quad \text{tr}(C_k W) \leq b_k, \quad k \in K.
\end{align*}
\]

(3)

- Relaxed problem is convex.
- Semidefinite programming.
- $\text{Rank}(W) = 1$ constraint is ignored.

If rank $W_* = 1$ then $W_*$ has a unique decomposition $W_* = x_* x_*^H$, where

\[
r_* = \text{tr}(CW_*) = x_*^H C x_* = p_*.
\]
Dual Problem

Dual problem $DP$:

\[
\begin{align*}
\text{maximize} \quad &\sum_{k \in K} \lambda_k b_k \\
\text{subject to} \quad &C + \sum_{k \in K} \lambda_k C_k \geq 0.
\end{align*}
\]

- If condition holds, and $\lambda \gg 0$ then we have zero duality gap in relaxed problem.
Rank condition

If the resulting $W$ matrix has a rank more than 1,

Perturbed relaxed problem $R\!P^\varepsilon$:

\[
\text{minimize} \quad \sum_{W \succeq 0} \text{tr}(CW) - \varepsilon \sum_{k \in K} \left[ b_k - \text{tr}(C_kW) \right]
\]

subject to:
\[
\text{tr}(C_kW) \leq b_k, \quad k \in K.
\]

Perturbed dual problem $D\!P^\varepsilon$:

\[
\text{maximize} \quad -\sum_{k \in K} \lambda_k b_k
\]

subject to:
\[
A(\lambda) \succeq 0, \quad \lambda_k \geq \varepsilon, \quad k \in K.
\]
Steps to the Solution

- Solve Relaxed Problem
  - If $\text{rank}(W^*) = 1$, then $r^* = p^*$
  - If $\text{rank}(W^*) > 1$, then choose $\varepsilon$ and model Perturbed Relaxed Problem
- Solve $\text{RP}^\varepsilon$ and resulting $\text{rank}(W^\varepsilon)$ is always $\leq 1$
- Decompose $W^\varepsilon = x^\varepsilon x^\varepsilon^T$
- $x^\varepsilon$ is a feasible point, but may not be optimal
Optimal Power Flow Problem

**Optimal power flow problem** $OPF$:

\[
\begin{align*}
& \text{minimize} \quad V^H CV \\
& \text{subject to:} \\
& P_k \leq V^H \Phi_k V \leq \overline{P}_k, \quad k \in [n], \\
& Q_k \leq V^H \Psi_k V \leq \overline{Q}_k, \quad k \in [n], \\
& W_k \leq V^H J_k V \leq \overline{W}_k, \quad k \in [n], \\
& V^H M_{ij} V \leq \overline{F}_{ij}, \quad i \sim j, \\
& V^H T_{ij} V \leq \overline{L}_{ij}, \quad i \sim j,
\end{align*}
\]

(17a) \quad (17b) \quad (17c) \quad (17d) \quad (17e)

The thermal loss of the line connecting buses $i$ and $j$ is

\[
L_{ij} = L_{ji} = P_{ij} + P_{ji} = V^H T_{ij} V
\]

where $T_{ij} = T_{ji} := M_{ij} + M_{ji} \geq 0$. 
Complex Domain

For a given tree graph and for a fixed line (i, j) in tree.

\[(a) \ [\Phi_i]_{ij} = -g_{ij}/2 + ib_{ij}/2,\]
\[(b) \ [\Phi_j]_{ij} = -g_{ij}/2 - ib_{ij}/2,\]
\[(c) \ [\Psi_i]_{ij} = -b_{ij}/2 - ig_{ij}/2,\]
\[(d) \ [\Psi_j]_{ij} = -b_{ij}/2 + ig_{ij}/2,\]
\[(e) \ [M^{ij}]_{ij} = -g_{ij}/2 + ib_{ij}/2,\]
\[(f) \ [M^{ji}]_{ij} = -g_{ij}/2 - ib_{ij}/2,\]
\[(g) \ [T^{ij}]_{ij} = [T^{ji}]_{ij} = -g_{ij}.\]
Complex Plane Representation

- For a fixed branch \((i,j)\);
Heuristic Approach

1) For $k \in \mathcal{K}$, linearize the function $f_k(x) = x^H C_k x$ around the point $x_m$ and call this function $f_k^{(m)}(x)$, i.e.,

$$f_k^{(m)}(x) = x_m^H C_k x_m + 2 \text{Re} \left[ x_m^H C_k (x - x_m) \right].$$

2) For $k \in \mathcal{K}$, define

$$s_k^{(m)}(x) := \begin{cases} 
  b_k - f_k^{(m)}(x), & \text{if } f_k^{(m)}(x) \leq b_k, \\
  0 & \text{if } b_k \leq f_k^{(m)}(x) \leq \bar{b}_k, \\
  f_k^{(m)}(x) - \bar{b}_k, & \text{if } \bar{b}_k \leq f_k^{(m)}(x).
\end{cases}$$

We can interpret $s_k^{(m)}(x)$ as the amount by which the linearized function $f_k^{(m)}$ violates the inequality constraint $b_k \leq f_k^{(m)}(x) \leq \bar{b}_k$. 
Heuristic Approach (cont.)

3) Compute $x_{m+1}$ using

$$x_{m+1} = \arg \min_{x \in \mathbb{C}^n} \sum_{k \in \mathcal{K}} [s_k^{(m)}(x)]^2$$

subject to: $\|x - x_m\|_1 \leq \gamma$,

where $\|\cdot\|_1$ denotes the $\ell^1$ norm and $\gamma$ is the maximum allowable step-size. This is a parameter for the algorithm and should be chosen such that the linearization $f_k^{m}(x)$ is a reasonably good approximation of the quadratic form $f_k(x)$ for all $k \in \mathcal{K}$ in the $\ell^1$ ball centered around $x_m$ with radius $\gamma$.

4) If $x_{m+1}$ satisfies all constraints in $P$, then the algorithm ends with $\tilde{x} = x_{m+1}$.

- No optimality, only having possibility to get a feasible point.
Sub-optimality Value

described above is used to find a feasible point of $OPF$. The feasible point obtained may not be optimal for the original problem, so we characterized its sub-optimality by defining the following quantity.

$$\eta := \frac{\text{Objective value at heuristically reached feasible point}}{\text{Objective value at optimal point of relaxed problem}} - 1.$$  

- Only loss minimization or voltage norm minimization are considered as an objective function in the results.