Modification in Factor

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• Modification of Cholesky Factor

• Modification in Newton method
Modification of Cholesky Factor

• Consider the case where a symmetric positive definite matrix $A$ is modified by a symmetric matrix of rank 1.

$$\bar{A} = A + \alpha zz^T$$

• Assuming that the Cholesky factors of $A$ are known: $A = LDL^T$, we wish to determine the factors,

$$\bar{A} = \bar{L}\bar{D}\bar{L}^T$$
• $\tilde{A} = A + \alpha zz^T = L(D + \alpha pp^T)L^T$

$\Rightarrow D + \alpha pp^T = \tilde{L}\tilde{D}\tilde{L}^T$
Method 1: Using Classical Cholesky Factorization

\[
\begin{pmatrix}
1 & \tilde{l}_{21} \\
\tilde{l}_{21} & \tilde{L}_{22}
\end{pmatrix}
\begin{pmatrix}
\tilde{d}_1 \\
\tilde{D}_2
\end{pmatrix}
\begin{pmatrix}
1 & \tilde{l}_{12} \\
\tilde{L}_{22}^T & \tilde{D}_2^T
\end{pmatrix}
= 
\begin{pmatrix}
d_1 \\
D_2
\end{pmatrix}
+ 
\begin{pmatrix}
\alpha p_1^2 & \alpha p_1 p_2^T \\
\alpha p_1 p_2 & \alpha p_2^T
\end{pmatrix}
\]
• $\tilde{d}_1 = d_1 + \alpha p_1^2$

• $\tilde{l}_{21} = \frac{\alpha p_1}{\tilde{d}_1} p_2$

• $\tilde{L}_{22} \tilde{D}_{22} \tilde{L}_{22}^T = D_{22} + \alpha p_2 p_2^T - \tilde{d}_1 \tilde{l}_{21} \tilde{l}_{21}^T$

  $$= D_{22} + (\alpha - \frac{\alpha^2 p_1^2}{\tilde{d}_1}) p_2 p_2^T = D_{22} + \frac{\alpha d_1}{\tilde{d}_1} p_2 p_2^T$$
\[ \tilde{d}_1 = d_1 + \alpha p_1^2 \]

- If \( \alpha < 0 \) and \( \tilde{A} \) is near to singularity, it is possible that rounding error could cause the diagonal element \( d \) to become zero or arbitrarily small. In such cases, it is also possible that the \( d \) could change sign.
• Method 2:

\[ \tilde{A} = LD^{1/2}(I + \alpha vv^T)D^{1/2}L \]

\[ I + \alpha vv^T = \tilde{L}\tilde{L}^T \text{ (diagonal elements in } \tilde{L} \text{ is not 1)} \]

\[ \tilde{A} = LD^{1/2}\tilde{L}\tilde{L}^T D^{1/2}L \]
\[ I + \alpha vv^T = \tilde{L}\tilde{L}^T \]

\[
\begin{pmatrix}
  l_{11} & l_{12} \\
  l_{21} & L_{22}
\end{pmatrix}
\begin{pmatrix}
  l_{11} & l_{12} \\
  L_{22}^T
\end{pmatrix} =
\begin{pmatrix}
  1 + \alpha v_1^2 & \alpha v_1 v_2^T \\
  \alpha v_1 v_2 & 1 + \alpha v_2 v_2^T
\end{pmatrix}
\]

\[ l_{11} = \sqrt{1 + \alpha v_1^2} \]

\[ l_{21} = \frac{\alpha v_1}{l_{11}} v_2 \]

\[ L_{22}L_{22}^T = I + \alpha v_2 v_2^T - l_{21}l_{21}^T = I + \left( \alpha - \frac{\alpha^2 v_1^2}{l_{11}^2} \right) v_2 v_2^T = I + \frac{\alpha}{l_{11}^2} v_2 v_2^T \]
Modification in Newton method

• Newton method for approximating a root to $f(x)=0$ is given by the iteration:

$$ J(x_k)(x_{k+1} - x_k) = -f(x_k) $$

• We attempt to approximate the Jacobian using rank one corrections:

$$ J'(x_{k+1}) = J(x_k) - u_k v_k^T $$
$J'(x_{k+1})$ is chosen so that:

$$J'(x_{k+1})(x_{k+1} - x_k) = f(x_{k+1}) - f(x_k)$$

$$\rightarrow -u_k v_k^T (x_{k+1} - x_k) = f(x_{k+1})$$
\[ -u_k v_k^T (x_{k+1} - x_k) = f(x_{k+1}) \]

- Q1: how to keep sparsity?
- Q2: how to keep stability?
Thank you